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A Markovian random coupling model for turbulence

By U. FRISCH, M. LESIEUR
Centre National de la Recherche Scientifique, Observatoire de Nice, France
AND A. BRISSAUD
Ecole Nationale Supérieure de l'Aéronautique et de l'Espace, Toulouse, France

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The Markovian random coupling (MRC) model is a modified form of the stochastic model of the Navier–Stokes equations introduced by Kraichnan (1958, 1961). Instead of constant random coupling coefficients, white-noise time dependence is assumed for the MRC model. Like the Kraichnan model, the MRC model preserves many structural properties of the original Navier–Stokes equations and should be useful for investigating qualitative features of turbulent flows, in particular in the limit of vanishing viscosity. The closure problem is solved exactly for the MRC model by a technique which, contrary to the original Kraichnan derivation, is not based on diagrammatic expansions. A closed equation is obtained for the functional probability distribution of the velocity field which is a special case of Edwards' (1964) Fokker–Planck equation; this equation is an exact consequence of the stochastic model whereas Edwards' equation constitutes only the first step in a formal expansion based directly on the Navier–Stokes equations. From the functional equation an exact master equation is derived for simultaneous second-order moments which happens to be essentially a Markovianized version of the single-time quasi-normal approximation characterized by a constant triad-interaction time.

The explicit form of the MRC master equation is given for the Burgers equation and for two- and three-dimensional homogeneous isotropic turbulence.

1. Introduction

Stochastic models as a tool for incompressible turbulence theory were introduced by Kraichnan (1958, 1961). They have the very interesting feature that they embody many of the structural properties of the Navier–Stokes equations: the same nonlinearity and dimensionality, non-local pressure interactions, conservation of total energy and total helicity (Brissaud, Frisch, Leorat, Lesieur & Mazure 1973) by the nonlinear terms, the existence of absolute equilibrium solutions for the inviscid truncated equations (Lee 1952; Kraichnan 1967, 1973; Frisch *et al.* 1973), etc. At the same time these models lead, in a certain asymptotic limit, to a closed set of 'master equations' for the two-time velocity covariance and response function.

For the first stochastic model introduced, the random coupling model, the original derivation (Kraichnan 1961) makes use of formal diagrammatic perturbation expansions which are known to diverge. Since then, simpler Markovian models have been introduced (Leith 1971; Kraichnan 1971; Herring & Kraichnan 1972). The corresponding master equations have been related to linear Langevin-type models but no direct derivation from a nonlinear stochastic model of the Navier–Stokes equations is to be found in the literature.

It is our purpose to give a new self-contained non-diagrammatic derivation for a very simple stochastic model called the Markovian random coupling (MRC) model. This derivation is based on a Liouville equation formalism used by Herring (1965, 1966) and Edwards (1964), here applied to a stochastic model of the Navier–Stokes equations and not to the original equations. A master equation is obtained for the simultaneous velocity covariance which does not involve the response function. We mention that a non-diagrammatic derivation of the Kraichnan equation has been given previously by Frisch & Bourret (1970) by the method of parastochastic operators; however, this technique applies only to linear stochastic equations (e.g. turbulent diffusion) with no obvious generalization to nonlinear equations.

2. The Markovian random coupling model: derivation of the master equation

In this section we are concerned with statistical solutions of the Navier–Stokes equations; in order to ensure the existence of stationary solutions we add a random driving term $\mathbf{f}(\mathbf{x},t)$. To construct the Markovian random coupling (MRC) model it is convenient to write the Navier–Stokes equations in a more abstract form. For this purpose, we first eliminate the pressure term in the usual way assuming, for simplicity, that there are no boundary conditions at finite distances; this yields

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) = -(\mathbf{u}\,.\,\nabla)\,\mathbf{u} - \nabla\left\{\frac{1}{4\pi|x|}*\nabla\,.\,(\mathbf{u}\,.\,\nabla)\,\mathbf{u}\right\} + \nu\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{x},t), \tag{2.1}$$

where $*$ denotes a convolution product in x space. This equation can be written in a more abstract form as

$$\partial u/\partial t = L(u,u) + Du + f, \tag{2.2}$$

where u and f are elements in a function space and D and L are respectively linear and bilinear operators. If a basis is chosen in this function space (e.g. a Fourier basis), the above equation will take the form

$$\partial u_a(t)/\partial t = L_{abc}u_b u_c + D_{ab}u_b + f_a(t), \tag{2.3}$$

where *the indices a, b and c stand both for the space components of vectors and for the Fourier wave vectors.* Since $L_{abc}u_b u_c$ does not change if we symmetrize L_{abc} with respect to b and c, we assume $L_{abc} = L_{acb}$. With this notation, the MRC equations read

$$\partial u_a^\alpha(t)/\partial t = N^{-1}\Phi_{\alpha\beta\gamma}(t)\,L_{abc}u_b^\beta u_c^\gamma + D_{ab}u_b^\alpha + f_a^\alpha(t). \tag{2.4}$$

The indices α, β and γ run from 1 to N; eventually we shall let $n \to \infty$. For fixed α, β and γ, the coupling coefficient $\Phi_{\alpha\beta\gamma}(t)$ is a real Gaussian white-noise process with zero mean and covariance

$$\langle\Phi(t)\,\Phi(t')\rangle = \tau_0\delta(t-t'), \tag{2.5}$$

where the parameter τ_0 has the dimensions of time and will be called the triad-interaction time. Furthermore the various $\Phi_{\alpha\beta\gamma}$ are identically distributed and independent, with the restriction that $\Phi_{\alpha\beta\gamma}$ must be completely symmetric with respect to α, β and γ to ensure energy conservation. The $f_a^\alpha(t)$ constitute a set of N identically distributed and independent Gaussian driving forces with zero mean and covariance

$$\langle f_a^\alpha(t) f_b^\alpha(t')\rangle = \tau_0 F_{ab}\,\delta(t-t') \quad \text{(no summation over } \alpha\text{)}, \tag{2.6}$$

where F_{ab} is an arbitrary space covariance which controls the energy injection. Finally, the random initial velocity fields $u_a^\alpha(0)$ are given, identically distributed, independent among themselves and independent of the driving forces and coupling coefficients.

It must be stressed that the greek superscripts differ basically from the latin subscripts: the latin indices stand for both the space components of vectors and for Fourier wave vectors, whereas the greek indices label various realizations of the turbulent flow which are coupled through the $\Phi_{\alpha\beta\gamma}$. The MRC model differs from the Kraichnan (1961) random coupling model only by the introduction of time-dependent coupling coefficients. Similar 'Markovian' models are discussed by Herring & Kraichnan (1972).

The MRC model can be viewed as a random coupling introduced among a large collection of identically distributed independent turbulent flows. The original problem already contains two stochastic elements: the initial velocity field and the driving force. The introduction of an additional stochastic element with no memory (Markovian) allows solution of the closure problem in the limit $N \to \infty$. Indeed we shall show that the simultaneous covariance of the model

$$U_{ab}(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} \langle u_a^\alpha(t)\, u_b^\alpha(t)\rangle \tag{2.7}$$

satisfies a closed master equation. Notice that the MRC model preserves the structural properties listed in § 1.

Now, we turn to the derivation of the MRC master equation for the simultaneous velocity covariance. To avoid the use of functional differentiation, let us assume for convenience that all the indices in (2.3) are discrete. This equation can be viewed as describing the motion of a point with co-ordinates u_a^α in a certain phase space. For each realization of the driving forces $f_a^\alpha(t)$ and the coupling coefficients $\Phi_{\alpha\beta\gamma}(t)$ we consider a Gibbsian ensemble of initial velocity fields characterized by a density $\rho(0; u_a^\alpha)$ in the phase space. When each realization evolves according to (2.4), the density changes according to the Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial u_a^\alpha}(\dot{u}_a^\alpha \rho) = 0. \tag{2.8}$$

Equation (2.8) is just an equation of continuity in the phase space. It is easily seen that this equation is the functional Fourier transform of the Hopf (1962) equation for the characteristic functional. For the ordinary Navier–Stokes equations a similar Liouville formalism has been used by Herring (1965) and Edwards (1964). In full, the Liouville equation reads

$$\frac{\partial \rho}{\partial t} + \frac{1}{N}\Phi_{\alpha\beta\gamma}(t)\, L_{abc} \frac{\partial}{\partial u_a^\alpha}(u_b^\beta u_c^\gamma \rho) + \nu D_{ab} \frac{\partial}{\partial u_a^\alpha}(u_b^\alpha \rho) + f_a^\alpha(t) \frac{\partial \rho}{\partial u_a^\alpha} = 0. \tag{2.9}$$

We shall be interested in the probability density $P(t; u_a^\alpha)$ defined as the average of ρ over the random driving force and coupling coefficients. In order to obtain an equation for $P = \langle \rho \rangle$ we notice that (2.9) is a linear stochastic equation of the form

$$\partial \rho / \partial t = L_0 \rho + L(t) \rho, \tag{2.10}$$

where the operator L_0, corresponding to the viscosity term, is a deterministic operator and $L(t)$, corresponding to the nonlinear and driving terms, is a stochastic operator with white-noise time dependence, i.e.

$$\langle L(t) \rangle = 0,$$
$$\langle L(t) \, L(t') \rangle = \tau_0 \delta(t-t') \, M.$$

Now, it is known that the mean solution of (2.10) satisfies a diffusion equation which reads (Kubo 1963; Leibowitz 1963; Brissaud & Frisch 1974)

$$\partial \langle \rho \rangle / \partial t = L_0 \langle \rho \rangle + \tfrac{1}{2} \tau_0 M \langle \rho \rangle. \tag{2.11}$$

A similar remark was used by Edwards (1964) with the difference that his Liouville equation had white-noise forcing terms but no white-noise coupling coefficients. Applying (2.11) to (2.9) and using the statistical properties of the driving forces and coupling coefficients, we obtain after some algebra

$$\begin{aligned}\frac{\partial P}{\partial t} = & -\nu D_{ab} \frac{\partial}{\partial u_a^\alpha} (u_b^\alpha P) + \frac{\tau_0}{N^2} L_{abc} L_{a'b'c'} \frac{\partial}{\partial u_a^\alpha} \left(u_b^\beta u_c^\gamma \frac{\partial}{\partial u_{a'}^\alpha} (u_{b'}^\beta u_{c'}^\gamma P) \right) \\ & + \frac{2\tau_0}{N^2} L_{abc} L_{a'b'c'} \frac{\partial}{\partial u_a^\alpha} \left(u_b^\beta u_c^\gamma \frac{\partial}{\partial u_{a'}^\beta} (u_{b'}^\alpha u_{c'}^\gamma P) \right) + \frac{\tau_0}{2} F_{aa'} \frac{\partial^2}{\partial u_a^\alpha \partial u_{a'}^\alpha} P. \end{aligned} \tag{2.12}$$

Our purpose is now to show that in the limit $N \to \infty$ equation (2.12) admits factorized solutions, i.e. that the different modes decouple asymptotically.† For this, we introduce the reduced probability densities $P_S(t; u^1, \ldots, u^S)$ obtained from P by integration over all the u^α with $\alpha > S$. The same integrations applied to (2.12) lead to the following set of equations for the evolution of the reduced probability densities:

$$\begin{aligned}\frac{\partial}{\partial t} P_S(t; u^1, \ldots, u^S) = & \sum_{\alpha=1}^{S} \left\{ \frac{\tau_0}{2} F_{aa'} \frac{\partial^2}{\partial u_a^\alpha \partial u_{a'}^\alpha} P_S - \nu D_{ab} \frac{\partial}{\partial u_a^\alpha} (u_b^\alpha P_S) \right\} \\ & + \int \frac{1}{N^2} \sum_{\alpha, \beta, \gamma} C(u^\alpha, u^\beta, u^\gamma) \, P \, du^{S+1} \ldots du^N, \end{aligned} \tag{2.13}$$

where the operators $C(u^\alpha, u^\beta, u^\gamma)$ are defined by

$$C(u^\alpha, u^\beta, u^\gamma) \Psi = \tau_0 L_{abc} L_{a'b'c'} \frac{\partial}{\partial u_a^\alpha} \left\{ u_b^\beta u_c^\gamma \left[\frac{\partial}{\partial u_{a'}^\alpha} (u_{b'}^\beta u_{c'}^\gamma \Psi) + 2 \frac{\partial}{\partial u_{a'}^\beta} (u_{b'}^\alpha u_{c'}^\gamma \Psi) \right] \right\}$$

for any Ψ. At this point we notice that, from the assumptions made in the preceding sections, the probability density P is invariant under permutations of the different u^α. Using this symmetry property and separating the various terms

† Our factorization property, which concerns only (greek) superscripts, is an exact consequence of the stochastic model and has nothing to do with the factorization assumptions of Herring (1966), which concern latin subscripts.

in the integral on the right-hand side of (2.13) according to the respective values of α, β, γ and S, we find that, for $N \to \infty$, only the terms with $\alpha \leqslant S$ and $\beta, \gamma, > S$ have a non-zero limit; this leads to the following hierarchy for the limiting reduced probability densities $P_S^\infty = \lim_{N\to\infty} P_S$:

$$\frac{\partial}{\partial t} P_S^\infty (t; u^1, \ldots, u^S) = \sum_{\alpha=1}^{S} \left\{ \frac{\tau_0}{2} F_{aa'} \frac{\partial^2}{\partial u_a^\alpha \, \partial u_{a'}^\alpha} P_S^\infty - \nu D_{ab} \frac{\partial}{\partial u_a^\alpha} (u_b^\alpha P_S^\infty) \right\} + \sum_{\alpha=1}^{S} \int C(u^\alpha, u^{S+1}, u^{S+2}) \, P_{S+2}^\infty \, du^{S+1} \, du^{S+2}. \quad (2.14)$$

This derivation parallels the standard derivation of the BBGKY hierarchy from the Liouville equation obtained in the thermodynamic limit in statistical mechanics (Prigogine 1962, chap. 7). The remarkable point about the linear equation (2.14) is that it admits factorized solutions. Indeed, setting

$$P_S^\infty(t; u^1, \ldots, u^S) = p(t; u^1) \ldots p(t; u^S), \quad (2.15)$$

we readily find that (2.14) is identically satisfied provided that p satisfies the following nonlinear equation (the analogue of the Boltzmann equation of kinetic theory):

$$\frac{\partial}{\partial t} p(t; u^1) = \frac{\tau_0}{2} F_{aa'} \frac{\partial^2}{\partial u_a^1 \, \partial u_{a'}^1} p - \nu D_{ab} \frac{\partial}{\partial u_a^1} (u_b^1 p) + \iint C(u^1, u^2, u^3) \, p(t; u^1) \, p(t; u^2) \, p(t; u^3) \, du^2 \, du^3. \quad (2.16)$$

Since, by the assumption of initial statistical independence of the u^α, the probability densities are factorized initially, they will remain so at any later time. Notice also that (2.16), although written as a partial differential equation, is in fact a functional differential equation since the Fourier variables are continuous.

The nonlinear 'Boltzmann' equation (2.16) appears, at first sight, even more formidable than the Hopf equation written directly for the Navier–Stokes equations. The main difference is that the Hopf equation does not lead to a closed equation for the simultaneous velocity covariance whereas, from (2.16), we derive the following master equation:

$$\partial U_{dd'}/\partial t = \tau_0 F_{dd'} + \nu D_{d'b} U_{db} + \nu D_{db} U_{bd'} + 2\tau_0 L_{dbc} L_{d'b'c'} U_{bb'} U_{cc'} + 2\tau_0 L_{dbc} L_{bb'c'} U_{cc'} U_{d'b'} + 2\tau_0 L_{d'bc} L_{bb'c'} U_{db'} U_{cc'}, \quad (2.17)$$

where the covariance $U_{dd'}$ is defined by

$$U_{dd'} = \int u_d^1 u_{d'}^1 \, p(t; u^1) \, du^1.$$

Equation (2.17) is obtained from (2.16) after multiplication by $u_d^1 \, u_{d'}^1$, and integration over u^1. Furthermore, it may be checked that the stationary solution of (2.16) is a Gaussian probability distribution with zero mean and covariance given by the solution of (2.17). We have thus proved that in the MRC model the various modes decouple and become asymptotically Gaussian. The Gaussian property is peculiar to the MRC model and does not hold for the Navier–Stokes equations

(Ogura 1963). Finally, the master equation (2.17) can be rewritten in a more compact form. Let the Navier–Stokes equations or any other nonlinear quadratic equation be written in some function space as

$$\partial u(t)/\partial t = L(u(t), u(t)) + L_0 u(t) + f(t),$$

where L_0 and L are respectively linear and symmetric bilinear time-independent operators. Let $f(t)$ be random with zero mean and covariance

$$\langle f(t) \otimes f(t') \rangle = \tau_0 \delta(t - t')\, F,$$

where $\otimes$ denotes a tensor product. Then, the MRC master equation reads

$$\frac{\partial}{\partial t}\langle u \otimes u \rangle = \tau_0 F + \langle L_0 u \otimes u \rangle + \langle u \otimes L_0 u \rangle + 2\tau_0 L(u, u) \otimes L(u, u)$$
$$+ 2\tau_0 L(L(u, u), u) \otimes u + 2\tau_0 u \otimes L(u, L(u, u)), \quad (2.18)$$

where u factors belonging to the same covariance have been linked together.

The master equation (2.18) is not new in many respects. First, it can be viewed as a Markovianized form of the single-time quasi-normal approximation (Tatsumi 1957) obtained by replacing a time integration merely by an instantaneous factor $\frac{1}{2}\tau_0$. Of course (2.18) does not lead to negative energy spectra, since we have shown that it is based on a stochastic model. Second, this equation may be viewed as a special case of an *approximate* equation obtained by Edwards (1964) by Fokker–Planck techniques (cf. also Edwards & McComb 1969); it turns out that for the MRC model the expansion terminates exactly at its first term. Third, (2.18) is also a Markovianized form of the Kraichnan (1959) DIA equations. Finally, (2.18) can be easily derived from a *linear* white-noise Langevin model of the form considered by Leith (1971) and Kraichnan (1971). The basic result of this section is that we have derived the master equation (2.18) from a *nonlinear* stochastic model of the Navier–Stokes equations [equation (2.4)]. The present derivation is the first one not based on diagrammatic techniques.

3. Explicit forms of the master equation

In §2, we have chosen rather general notation so as to make the MRC master equations directly applicable to any quadratically nonlinear evolution problems such as the MHD equations, the Boussinesq equations, the Vlasov equation, etc. The MRC master equation takes its simplest form when applied to the Burgers equation (Burgers 1950).

$$\frac{\partial}{\partial t} u(x,t) + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad (3.1)$$

assuming $\langle u \rangle = 0$ and homogeneity, and defining

$$U(x,t) = \lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} \langle u^\alpha(y,t)\, u^\alpha(y+x,t) \rangle, \quad (3.2)$$

$$\langle f^\alpha(y,t) f^\beta(y+x,t') \rangle = \tau_0 \delta_{\alpha\beta} \delta(t-t') F(x). \quad (3.3)$$

The master equation (2.17) now reads

$$\frac{\partial}{\partial t} U(x,t) = \tau_0 F(x) + 2\nu \frac{\partial^2}{\partial x^2} U(x,t) - \frac{\tau_0}{2} \frac{\partial^2}{\partial x^2} [U(x,t) - U(0,t)]^2. \tag{3.4}$$

This equation can be solved exactly for the stationary case (Lesieur 1973). The non-stationary case is very easily solved numerically (Brissaud, Frisch, Leorat, Lesieur, Mazure, Pouquet, Sadourny & Sulem 1973). It has also been studied from a mathematical viewpoint by Brauner, Penel & Teman (1974).

For three-dimensional homogeneous isotropic non-helical turbulence, the MRC master equation reads

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = \tau_0 F(k) + \frac{\tau_0}{4} \iint_{\Delta_k} \frac{k}{pq} b_3(k,p,q) \times \{k^2 E(p,t) E(q,t) - p^2 E(q,t) E(k,t)\} dp\, dq. \tag{3.5}$$

$E(k,t)$ is the usual energy spectrum such that

$$\langle u^2(t)\rangle = \int_0^\infty E(k,t)\, dk,$$

$F(k)$ is the injection spectrum and the integration in the p, q plane is over the domain Δ_k such that k, p and q can be the sides of a triangle. The coefficient $b_3(k,p,q)$ is given by

$$b_3(k,p,q) = (p/k)(xy + z^3), \tag{3.6}$$

where x, y and z are the cosines of the interior angles of the k, p, q triangle

For two-dimensional homogeneous isotropic turbulence, we obtain with the same notation

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = \tau_0 F(k) + \frac{\tau_0}{2\pi} \iint_{\Delta_k} \frac{k^2}{pq} b_2(k,p,q) \times \{k E(p,t) E(q,t) - p E(q,t) E(k,t)\} dp\, dq, \tag{3.7}$$

where

$$b_2(k,p,q) = 2\frac{p}{k}\left\{\frac{xy - z + 2z^3}{(1-x^2)^{\frac{1}{2}}}\right\}. \tag{3.8}$$

It is easily shown (e.g. by dimensional inspection) that the MRC energy inertial range is characterized by a k^{-2} spectrum. This follows from the constancy of the triad interaction time τ_0 over the inertial range and can easily be modified by introducing a suitable triad-dependent interaction time τ_{kpq} as was done by Orszag (1969) and Kraichnan (1971).

Although it may be tempting to make such a change in order to ensure more quantitative contact with turbulent flows, this is probably not necessary for many qualitative investigations. Problems such as the onset of dissipation after a finite time in inviscid three-dimensional turbulence (Brissaud, Frisch, Leorat, Lesieur, Mazure, Pouquet, Sadourny & Sulem 1973; Frisch 1973), the existence of direct enstrophy and inverse energy cascades in two-dimensional turbulence (Pouquet, Lesieur & André 1973) and the existence of helicity cascades in three-dimensional isotropic turbulence lacking reflexional symmetry (Lesieur 1973) can all be tackled both by analytic and numerical methods using the MRC model, even at very high Reynolds numbers.

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